Individual Round Solutions

GLMT 2025

April 19, 2025

1. **[6]** The grid below is filled with the numbers 1, 2, 3, 4, 5, and 6, such that any two consecutive numbers are in adjacent cells. Find the number in the shaded cell.

	1	
3		(shaded)

Proposed by: Muztaba Syed

Solution. $\boxed{5}$

Note that this must be the last odd number, which is $\boxed{5}$. This can be achieved by putting 2 in the top left, and then going around the whole grid.

2	1	6
3	4	5

□

2. **[6]** Find

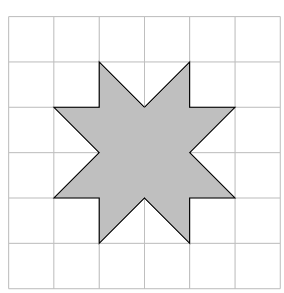
$$\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{25}}}}}}.$$

Proposed by: Muztaba Syed

Solution. $\boxed{5}$

Note that $5 = \sqrt{25} = \sqrt{20+\sqrt{25}}$, so all the terms will collapse into just $\boxed{5}$. □

3. **[6]** Ada colors a small star on graph paper (depicted below), where each small square has side length 1 unit. Find the area of the star.



Proposed by: Rohan Danda

Solution. $\boxed{8}$

This has a 2×2 square in the middle, along with 8 triangles of area $\frac{1}{2}$ attached to the sides. The area is then $2 \cdot 2 + \frac{1}{2} \cdot 8 = \boxed{8}$ □

4. **[6]** Find the median of the values

$$\frac{5}{7}, \frac{11}{13}, \frac{3}{5}, \frac{15}{17}, \frac{13}{15}.$$

Proposed by: Muztaba Syed

Solution. $\boxed{\dfrac{11}{13}}$

Note that the fractions are of the form $\frac{k-2}{k} = 1 - \frac{2}{k}$. When k is big, this will be closer to 1, so larger k will give larger fractions. Thus we need to find the median of $\{5, 7, 13, 15, 17\}$, which is 13. So the answer is $\boxed{\dfrac{11}{13}}$. □

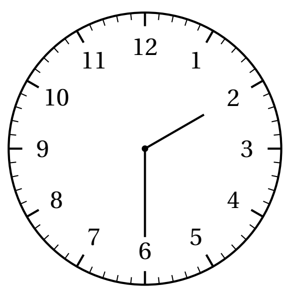
5. **[6]** Suppose A and B are digits such that $\overline{AB} + \overline{BA} = 99$. Find the maximum possible product of A and B.

Proposed by: Jacob Xu

Solution. $\boxed{20}$

We have $11A + 11B = 99$, so $A + B = 9$. If two things have a sum of 9, their maximum product is when they are close together. This gives us an answer of $4 \cdot 5 = 20$. □

6. **[7]** A clock is very slow, so it only travels 2 minutes every hour. If the clock is initially (correctly) set to 12:00 pm, find the time it shows when the time should be 2:30 pm on the same day.



Proposed by: Vedant Joshi

Solution. $\boxed{12:05}$

We see 2.5 hours pass, in which the clock only travels $2 \cdot 2.5 = 5$ minutes. So it will show $\boxed{12:05}$. □

7. **[7]** Find the last two digits of $20^{25} + 25^{20}$.

Proposed by: Atticus Oliver

Solution. $\boxed{25}$

Note that 20 is a multiple of 10, so raising it to a high power will give us a lot of trailing zeroes. Thus 20^{25} ends in 00. All powers of 25^{20} end with 25, so the answer is just $\boxed{25}$. □

8. **[7]** Find the number of real solutions x to the equation

$$x^2 = |2x|.$$

Proposed by: Peter Bai

Solution. $\boxed{3}$

If $x \geq 0$, then $2x \geq 0$ as well. We then have

$$x^2 = |2x| \implies x^2 = 2x \implies x^2 - 2x = 0 \implies x(x-2) = 0,$$

which gives us $x = 0, 2$ as possible solutions.

If $x < 0$, then $2x < 0$ as well and

$$x^2 = |2x| \implies x^2 = -2x \implies x^2 + 2x = 0 \implies x(x+2) = 0,$$

which gives us $x = -2$ as our last solution.

Our answer is thus $2 + 1 = \boxed{3}$. □

9. **[7]** Find

$$(1)\left(2+\frac{1}{1}\right)\left(3+\frac{1}{2+\frac{1}{1}}\right)\left(4+\frac{1}{3+\frac{1}{2+\frac{1}{1}}}\right)\left(5+\frac{1}{4+\frac{1}{3+\frac{1}{2+\frac{1}{1}}}}\right)\left(6+\frac{1}{5+\frac{1}{4+\frac{1}{3+\frac{1}{2+\frac{1}{1}}}}}\right).$$

Proposed by: Muztaba Syed

Solution. $\boxed{1393}$

We can systematically compute each of the 6 terms using the previous terms. The first is 1, the second is 3, the next is going to be $3 + \frac{1}{3} = \frac{10}{3}$. The fourth term will be 4 plus the reciprocal of the third term, which is $4 + \frac{3}{10} = \frac{43}{10}$. The fifth term will be 5 plus the reciprocal of the 4th, which is $5 + \frac{10}{43} = \frac{225}{43}$. Doing this again, the 6th term is $\frac{1393}{225}$. The answer is then

$$1 \cdot 3 \cdot \frac{10}{3} \cdot \frac{43}{10} \cdot \frac{225}{10} \cdot \frac{1393}{25} = \boxed{1393}.$$

□

10. **[7]** Right triangle ABC has $AB = 6$, $BC = 8$, and $\angle B = 90°$. A circle ω_1 of radius 3 is centered at A, and a circle ω_2 of radius 4 is centered at C. Find the largest possible distance between a point on ω_1 and a point on ω_2.

Proposed by: William Hua

Solution. $\boxed{17}$

Let the points on ω_1 and ω_2 be P and Q, respectively. By triangle inequality (technically quadrilateral inequality) on quadrilateral $PACQ$, $PQ \leq PA + AC + CQ$. The degenerate case occurs when P and Q lie on line AB but not segment AB. This gives an answer of $3 + \sqrt{6^2 + 8^2} + 4 = \boxed{17}$. □

11. **[8]** In the ground, there are 100 *Digletts* of heights 1 to 100. Three Digletts can form a *Dugtrio* if their heights form an increasing arithmetic sequence. A Diglett can be in multiple Dugtrios. Find the maximum number of distinct Dugtrios that can be formed with the 100 Digletts.

Proposed by: Ella Kim

Solution. $\boxed{2450}$

A Dugtrio consists of Digletts of height a, $a+n$, and $a+2n$ for integers $a, n > 0$. Notice that $a + 2n \leq 100$. This implies $a \leq 100 - 2n$, meaning a can take on $100 - 2n$ values. The maximum valid value of n is 49, giving the sum of a across these n as

$$\sum_{n=1}^{49} 100 - 2n,$$

which can be rearranged to

$$2\sum_{n=1}^{49} n = 2 \cdot \frac{49 \cdot 50}{2} = \boxed{2450}.$$

□

12. **[8]** A positive integer n has three digits in base nine and four digits in base four. Find the number of possible values of n.

Proposed by: William Hua

Solution. $\boxed{175}$

In base nine n is at least $100_9 = 81$, and at most $888_9 = 9^3 - 1 = 728$. In base four it is at least $1000_4 = 64$ and at most $3333_4 = 4^4 - 1 = 255$. Thus $81 \le n < 729$ and $64 \le n < 256$. So $81 \le n < 256$. The answer is $256 - 81 = \boxed{175}$. □

13. **[8]** Benicio puts the numbers 1 through 18 in the cells of a 3×6 grid. For each of the ten 2×2 grids within the 3×6 grid they write down the largest entry. Find the largest possible sum of the ten numbers Benicio writes.

Proposed by: Muztaba Syed

Solution. $\boxed{172}$

Each cell is in at most 4 of the 2×2 grids, so at most 4 of the grids can have maximum element 18. Similarly at most 4 can have maximum element 17. This leaves 2 grids with maximum element 16. The answer is then $18 \cdot 4 + 17 \cdot 4 + 16 \cdot 2 = \boxed{172}$.

This can be achieved with the following grid:

1	2	3	4	5	6
7	18	8	17	9	16
10	11	12	13	14	15

□

14. **[8]** Find the number of real solution pairs (x, y) that satisfy the following equations:

$$x^3 - x^2 = y^3 - y^2,$$
$$x^2 + y^2 = 1.$$

Proposed by: James Wu

Solution. $\boxed{4}$

By rearranging and factoring the first equation, we can get

$$x^3 - y^3 - x^2 + y^2 = 0 \Longrightarrow (x - y)(x^2 + xy + y^2 - x - y) = 0$$

If $x - y = 0$, by plugging it into the second equation we have two solutions:

$$(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

If $x - y \neq 0$, then $x^2 + xy + y^2 - x - y = 0$. Substituting $x^2 + y^2$ with 1 makes it

$$xy - x - y + 1 = 0 \Longrightarrow (x - 1)(y - 1) = 0$$

So either $x = 1$ or $y = 1$. Plugging each back into the second equation gives two solutions

$$(x, y) = (1, 0) \text{ and } (0, 1)$$

Therefore, the system of equations has $\boxed{4}$ solutions. □

15. **[8]** Bristopher Chrunner is writing a series of 2s and 3s on a blackboard. At one point, the product of all the numbers on the board is 162. When Bristopher Chrunner finishes writing, the sum of all the numbers on the board is 24. Find the number of ordered sequences of numbers that Bristopher Chrunner could have written on the board.

Proposed by: Atticus Oliver

Solution. $\boxed{35}$

When the product of the numbers on the board is 162, Bristopher Chrunner must have written one 2 and four 3s. There are $\binom{5}{1} = 5$ ways for Bristopher Chrunner to do this. This is a sum of 14 so far, so the remaining numbers Bristopher Chrunner wrote must sum to 10. This is done as either $2+2+2+2+2$ or $2+2+3+3$, which can be arranged $\binom{5}{5} = 1$ and $\binom{4}{2} = 6$ ways, respectively, for a total of 7 ways for the numbers after the product point, meaning there are $\boxed{35}$ total ways for Bristopher Chrunner to write the numbers on the board. □

16. **[9]** Find the sum of the roots of the equation

$$x^{25} + \left(\frac{1}{20} - x\right)^{25} = 0.$$

Proposed by: Isabella Li

Solution. $\boxed{\dfrac{3}{5}}$

This equation has 24 roots because the x^{25} terms cancel. Then because for each root r, $\frac{1}{20} - r$ is also a root, so there are 12 pairs of roots that sum to $\frac{1}{20}$. So the answer is $\frac{12}{20} = \boxed{\dfrac{3}{5}}$. □

17. **[9]** Two cubes have square bases $ABCD$ and $AEFG$ such that F, A, and C are collinear in that order and D and E lie on the same side of line segment FC. The difference in volumes of the two cubes is 240 and the difference in heights is 5. Find the area of hexagon $BCDEFG$.

Proposed by: Jacob Xu

Solution. $\boxed{48}$

Note that the volume difference is $a^3 - b^3$ and the difference in heights is $a - b$. The area of the hexagon is $a^2 + ab + b^2$, by splitting it up into 2 squares and 2 right triangles. But by difference of cubes $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, so the answer is $\frac{240}{5} = \boxed{48}$. □

18. **[9]** An ice cream cone with radius 2 and height $3\sqrt{2}$ has a hemisphere of radius 2 on top of it such that their bases coincide. Find the side length of the largest cube inscribed in the cone and the hemisphere such that every face of the cube is either parallel or perpendicular to the base of the cone.

Proposed by: Selena Ge

Solution. $\boxed{\dfrac{14\sqrt{2}}{9}}$

Consider the vertical cross section that goes through opposite vertices of the cube. Let the side length of the cube be $2s$, then similar triangles gives the distance from the bottom face of the cube to the apex of the cone is $3s$, so the distance from the bottom face of the cube to the base of the cone is $3\sqrt{2} - 3s$, and the distance from the top face to the base of the cone is $5s - 3\sqrt{2}$, so by Pythagorean theorem,

$$2^2 = 2s^2 + \left(5s - 3\sqrt{2}\right)^2.$$

Solving gives $s = \frac{7\sqrt{2}}{9}$, so the answer is $\boxed{\dfrac{14\sqrt{2}}{9}}$. □

19. **[9]** Alicia puts the letters of the word "TRIANGLES" into the cells of a 3×3 grid. Find the number of ways she can arrange the letters so that any two adjacent letters which are adjacent in the word TRIANGLES are in adjacent cells of the grid.

Proposed by: Selena Ge

Solution. $\boxed{40}$

We want to find the number of paths that go through all the squares of a 3×3 grid. If we checkerboard color the grid with the middle cell colored black, we can see that the first cell of the path must be in the center or one of the corners.

If it is in the center, the second cell can be in one of the four adjacent cells, then the rest of the path has two ways, either clockwise or counterclockwise.

If the first cell is a corner, the second has two ways, then the third cell can either be a corner or the center. If the third cell is a corner, there are 3 ways, and if it is the center, there is 1 way. This gives a total of $\boxed{40}$ ways. □

20. **[9]** Let $ABCDE$ be a regular pentagon of side length 1. Let $A'B'CD'E$ be the reflection of $ABCDE$ over CE. If the length of AB' is x, find x^2.

Proposed by: Sylvia Lee

Solution. $\boxed{\dfrac{7+\sqrt{5}}{2}}$

Clearly note that the angle at each point is $\frac{3\cdot 180^\circ}{5} = 108^\circ$. Define K to be the foot from A to CE. Since DEC is isosceles, it follows that $\angle DEC = 36^\circ$, or $\angle CEA = 72^\circ$. Thus, $AK = AE\sin 72^\circ = \sin 72^\circ$. It is well-known that $\sin 72^\circ = \sqrt{\frac{\sqrt{5}+5}{8}}$. Note that $AB \parallel CE$ as $\angle BAE = 108^\circ = 180^\circ - 72^\circ = 180^\circ - \angle AEC$. Similarly, $A'B' \parallel CE$. Also note that $AA' \perp CE$ since we reflected A across CE. Thus, $AB \perp AA'$, or $A'B' \perp AA'$. This implies that $AA' = 2AK = \sqrt{\frac{\sqrt{5}+5}{2}}$ Thus, by Pythagorean theorem:

$$\begin{aligned} AB'^2 &= AA'^2 + A'B'^2 \\ &= \frac{5+\sqrt{5}}{2} + 1 \\ &= \boxed{\frac{7+\sqrt{5}}{2}}. \end{aligned}$$

□